

THE SECOND HOMOTOPY GROUP IN TERMS OF COLORINGS OF LOCALLY FINITE MODELS AND NEW RESULTS ON ASPHERICITY

JONATHAN ARIEL BARMAN AND ELIAS GABRIEL MINIAN

ABSTRACT. We describe the second homotopy group of any CW-complex K by analyzing the universal cover of a locally finite model of K using the notion of G -coloring of a partially ordered set. As applications we prove a generalization of the Hurewicz theorem, which relates the homotopy and homology of non-necessarily simply-connected complexes, and derive new results on asphericity for two-dimensional complexes and group presentations.

1. INTRODUCTION

Every CW-complex has a *locally finite model*. This is a classical result of McCord [9, Theorem 3] who considered for any regular CW-complex K the space $\mathcal{X}(K)$ of cells of K with some specific topology, and defined a weak homotopy equivalence $\mu : K \rightarrow \mathcal{X}(K)$. The space $\mathcal{X}(K)$ can be viewed as a poset. The interaction between the topological and combinatorial nature of $\mathcal{X}(K)$ allows one to develop new techniques to attack problems of homotopy theory of CW-complexes (see [1]).

In this paper we use locally finite models to describe the second homotopy group of CW-complexes. The notion of G -coloring of a poset allows us to classify all the regular coverings of the space $\mathcal{X}(K)$. In particular, we obtain a description of the universal cover E of $\mathcal{X}(K)$ which is used to find an expression for the boundary map of a chain complex whose homology coincides with the singular homology of E . By the Hurewicz theorem and McCord's result, $\pi_2(K) = H_2(E)$. One of the applications of our description is the following result which reduces to the classical Hurewicz theorem when the complex is simply-connected.

Theorem 2.3. *Let K be a connected regular CW-complex of dimension 2 and let K' be its barycentric subdivision. Consider the full (one-dimensional) subcomplex L of K' spanned by the barycenters of the 1-cells and 2-cells. If the inclusion of each component of L in K' induces the trivial morphism between the fundamental groups, then $\pi_2(K) = \mathbb{Z}[\pi_1(K)] \otimes H_2(K)$.*

We also obtain results on asphericity of 2-complexes and group presentations. Recall that a connected 2-complex K is aspherical if $\pi_2(K) = 0$.

Theorem 3.1. *Let K be a 2-dimensional regular CW-complex and let K' be its barycentric subdivision. Consider the full (one-dimensional) subcomplex $L \subseteq K'$ spanned by the barycenters of the 2-cells of K and the barycenters of the 1-cells which are faces of exactly two 2-cells. Suppose that for every connected component M of L , $i_*(\pi_1(M)) \leq \pi_1(K')$*

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contains an element of infinite order, where $i_* : \pi_1(M) \rightarrow \pi_1(K')$ is the map induced by the inclusion. Then K is aspherical.

From this result one can deduce for example, the well-known fact that all compact surfaces different from S^2 and $\mathbb{R}P^2$ are aspherical.

To put our results in perspective, one should recall that it is an open problem, originally posted by Whitehead, whether any subcomplex of an aspherical 2-dimensional CW-complex is itself aspherical. We refer the reader to [5, 7, 13] for more details on Whitehead's asphericity question.

In Theorem 3.5 we prove a result on asphericity of group presentations which resembles the homological description of π_2 by Reidemeister chains (see [5, 10]).

2. COLORINGS AND A DESCRIPTION OF THE SECOND HOMOTOPY GROUP

A poset X will be identified with a topological space with the same underlying set as X and topology generated by the basis $\{U_x\}_{x \in X}$, where $U_x = \{y \in X \mid y \leq x\}$. If X and Y are posets, it is easy to see that a map $X \rightarrow Y$ is continuous if and only if it is order preserving. We denote by $\mathcal{K}(X)$ the simplicial complex whose simplices are the finite chains of the poset X (i.e. the classifying space of X). A result of McCord [9, Theorem 2] shows that there is a natural weak homotopy equivalence $\mathcal{K}(X) \rightarrow X$. Given a regular CW-complex K , its *face poset* $\mathcal{X}(K)$ is the poset of cells of K ordered by the face relation. Note that the classifying space of the poset $\mathcal{X}(K)$ is the barycentric subdivision of K and therefore, there is a weak homotopy equivalence $\mu : K \rightarrow \mathcal{X}(K)$. In particular the homology groups of the poset $\mathcal{X}(K)$ coincide with those of K . If $X = \mathcal{X}(K)$ is the face poset of a regular CW-complex, we can compute its homology by computing the cellular homology of K in the standard way (see [8, IX, §7]), namely, for each $n \geq 0$ let $C_n(X)$ be the free \mathbb{Z} -module generated by the points $x \in X$ of height $h(x) = n$. Recall that $h(x)$ is one less than the maximum number of points in a chain with maximum x . Choose for each edge (y, x) in the Hasse diagram of X a number $[x : y] \in \{1, -1\}$ in such a way that for every $x \in X$ of height 1,

$$\sum_{y \prec x} [x : y] = 0,$$

and for each pair $x, z \in X$ with $h(x) = h(z) + 2$,

$$\sum_{z \prec y \prec x} [x : y][y : z] = 0.$$

Here $y \prec x$ means that $y < x$ and there is no $y < z < x$. The differential $d : C_n(X) \rightarrow C_{n-1}(X)$ is defined by $d(x) = \sum_{y \prec x} [x : y]y$ in each basic element x . The homology of this chain complex is then the singular homology of the poset X (viewed as a topological space). The number $[x : y]$ is the incidence of the cell y in the cell x of K for certain orientations.

Suppose $p : E \rightarrow X = \mathcal{X}(K)$ is a topological covering ($\mathcal{X}(K)$ considered as a topological space). Using that p is a local homeomorphism it is easy to prove that E is also the space associated to a poset. Moreover, for each $x \in E$, $p|_{U_x} : U_x \rightarrow U_{p(x)}$ is a homeomorphism. In particular E is the face poset of a regular CW-complex ([4, Proposition 4.7.23]). If $y \prec x$ in E , $p(y) \prec p(x)$ in X . Given a choice of the incidences in X , we define $[x : y] = [p(x) : p(y)]$, which is a coherent choice for the incidences in E . Let $p : E \rightarrow \mathcal{X}(K)$ be a regular covering and let G be its group of deck (covering) transformations. Since G acts freely

on E and transitively on each fiber, $C_n(E) = \mathbb{Z}G \otimes C_n(X)$ is a free $\mathbb{Z}G$ -module with basis $\{x \in X \mid h(x) = n\}$. The differential $d : C_n(E) \rightarrow C_n(E)$ is a homomorphism of $\mathbb{Z}G$ -modules.

In [3] we characterized the regular coverings of locally finite posets (i.e. posets with finite U_x , for each x) in terms of *colorings*. We recall this result as it will be required in the description of the universal cover.

Let X be a locally finite poset. We denote by $\mathbf{E}(X)$ the set of edges in the Hasse diagram of X . An *edge-path* in X is a sequence $\xi = (x_0, x_1)(x_1, x_2) \dots (x_{k-1}, x_k)$ of edges, or opposites of edges. The set of closed edge-paths from a point $x_0 \in X$, with certain identifications and the operation given by concatenation, is a group $\mathcal{H}(X, x_0)$ naturally isomorphic to $\pi_1(X, x_0)$ (see [2] and [3]). This construction resembles the definition of the edge-path group of a simplicial complex. Given a group G , a G -*coloring* of a locally finite poset X is a map c which assigns a *color* $c(y, x) \in G$ to each edge $(y, x) \in \mathbf{E}(X)$. Given a G -coloring, if $y \prec x$ we define $c(x, y) = c(y, x)^{-1} \in G$. A G -coloring of X induces a *weight* map which maps an edge-path $\xi = (x_0, x_1)(x_1, x_2) \dots (x_{k-1}, x_k)$ to $w_c(\xi) = c(x_0, x_1)c(x_1, x_2) \dots c(x_{k-1}, x_k)$. A G -coloring c is said to be *admissible* if for any two chains $x = x_1 \prec x_2 \prec \dots \prec x_k = y$, $x = x'_1 \prec x'_2 \prec \dots \prec x'_l = y$ with same origin and same end, the weights of the edge-paths induced by the chains coincide. An admissible G -coloring c induces a homomorphism $W_c : \mathcal{H}(X, x_0) \rightarrow G$ which maps the class of a closed edge-path to its weight. The coloring c is *connected* if W_c is an epimorphism.

Two G -colorings c and c' of X are *equivalent* if there exists an automorphism $\varphi : G \rightarrow G$ and an element $g_x \in G$ for each $x \in X$ such that $c'(x, y) = \varphi(g_x c(x, y) g_y^{-1})$ for each $(x, y) \in \mathbf{E}(X)$.

Theorem 2.1. ([3, Corollary 3.5]) *Let X be a connected locally finite poset and let G be a group. There exists a correspondence between the set of equivalence classes of regular coverings $p : E \rightarrow X$ of X with $\text{Deck}(p)$ isomorphic to G and the set of equivalence classes of admissible connected G -colorings of X .*

Here $\text{Deck}(p)$ denotes the group of deck transformations of p . The covering associated to an admissible connected G -coloring c is the covering that corresponds to the subgroup $\ker(W_c)$ of $\mathcal{H}(X, x_0) \cong \pi_1(X, x_0)$. Theorem 3.6 of [3] tells us explicitly how to construct the covering $E(c)$ corresponding to c . It is the poset $E(c) = \{(x, g) \mid x \in X, g \in G\}$ with the relations $(x, g) \prec (y, gc(x, y))$ whenever $x \prec y$ in X . The covering map being the projection onto the first coordinate. The group G acts on $E(c)$ by left multiplication in the second coordinate.

Now, let K be a regular CW-complex and suppose c is any G -coloring of $X = \mathcal{X}(K)$ which corresponds to the universal cover, that is, c is an admissible and connected G -coloring such that $E = E(c)$ is simply-connected or, equivalently, $W_c : \mathcal{H}(X, x_0) \rightarrow G$ is an isomorphism. The second homotopy group of K is $\pi_2(K) = \pi_2(\mathcal{X}(K)) = H_2(E)$. The homology of E can be computed using the chain complex described above. In the case that K is two-dimensional, this computation is easier. In this case E is a poset of height two and $C_3(E) = 0$. A chain $\alpha \in C_2(E) = \mathbb{Z}G \otimes C_2(X)$ is a finite sum of the form

$$\alpha = \sum_{h(x)=2} \sum_{g \in G} n_g^x gx$$

where $n_g^x \in \mathbb{Z}$. The isomorphism between $C_2(E)$ and $\mathbb{Z}G \otimes C_2(X)$ identifies $(x, 1) \in E$ with x . Thus, $d : \mathbb{Z}G \otimes C_2(X) \rightarrow \mathbb{Z}G \otimes C_1(X)$ maps x to

$$\sum_{y \prec x} [x : y](y, c(y, x)^{-1}) = \sum_{y \prec x} [x : y]c(y, x)^{-1}y \in C_1(E) = \mathbb{Z}G \otimes C_1(X)$$

and then $d(\alpha) = \sum_{h(x)=2} \sum_{g \in G} \sum_{y \prec x} [x : y]n_g^x gc(y, x)^{-1}y$.

Therefore $\pi_2(K) = \ker(d)$ has the following description

$$\pi_2(K) = \left\{ \sum_{h(x)=2} \sum_{g \in G} n_g^x gx \mid \sum_{x \succ y} [x : y]n_{h \cdot c(y, x)}^x = 0 \ \forall y \in X \text{ with } h(y) = 1 \text{ and } \forall h \in G \right\}.$$

On the other hand, Theorem 4.4 and Remark 4.6 in [3] provide a concrete way to describe a coloring \hat{c} which corresponds to the universal cover. Let X be a locally finite poset and let D be a subdiagram(=subgraph) of the Hasse diagram of X . Suppose that the poset which corresponds to D is simply-connected and that D contains all the points of X (for instance, a spanning tree). Let G be the group generated by the edges $e \in \mathbf{E}(X)$ which are not in D with the following relations. For each pair of chains

$$x = x_1 \prec x_2 \prec \dots \prec x_k = y,$$

$$x = x'_1 \prec x'_2 \prec \dots \prec x'_l = y$$

with same origin and same end, we put a relation

$$\prod_{(x_i, x_{i+1}) \notin D} (x_i, x_{i+1}) = \prod_{(x'_i, x'_{i+1}) \notin D} (x'_i, x'_{i+1}).$$

According to Theorem 4.4 in [3] G is isomorphic to $\pi_1(X)$. Moreover, let \hat{c} be the G -coloring defined by $\hat{c}(e) = \bar{e}$, the class of e in G , for each $e \in \mathbf{E}(X)$. If $e \in D$, $\bar{e} = 1 \in G$. Then $W_{\hat{c}} : \mathcal{H}(X, x_0) \rightarrow G$ is an isomorphism, so \hat{c} corresponds to the universal cover of X . This coloring can be used in the formula above to compute $\pi_2(K)$.

Example 2.2. Consider the regular CW-complex K in Figure 1. It has three 0-cells, a, b, c , six 1-cells, q, r, s, t, u, v and four 2-cells, w, x, y, z . The Hasse diagram of $\mathcal{X}(K)$ appears in Figure 2. Let D be the subdiagram of the Hasse diagram given by the solid edges. It is easy to see that the space corresponding to D is simply connected because it is a contractible finite space(=dismantlable poset) [11, Section 4]. The group G generated by the dotted edges e_1, e_2, e_3, e_4, e_5 with relations $e_4 e_1 = 1, e_1 = e_5, e_2 = e_3, e_2 = e_5, e_3 = e_4$ is then isomorphic to the fundamental group of K . Hence $\pi_1(K) = \mathbb{Z}_2$.

For each $h \in G$ and each point of $\mathcal{X}(K)$ of height 1 we have one equation. These twelve equations describe $\pi_2(K)$. Denote γ the generator of G . Then $\bar{e}_1 = \bar{e}_2 = \gamma$. We can choose the incidences $[p_0 : p_1]$ according to the orientations of the cells in Figure 1. For instance, the equation corresponding to u and $h \in G$ is $0 = [w : u]n_{h \cdot \overline{uw}}^w + [y : u]n_{h \cdot \bar{e}_1}^y = n_h^w + n_{h\gamma}^y$. For each $h \in G$ the equations are

$$\begin{aligned} \text{Equations for } q : \quad & n_h^w + n_h^z = 0 \\ \text{Equations for } r : \quad & n_h^x + n_h^y = 0 \\ \text{Equations for } s : \quad & -n_h^w - n_h^x = 0 \\ \text{Equations for } t : \quad & -n_h^y - n_h^z = 0 \\ \text{Equations for } u : \quad & n_h^w + n_{h\gamma}^y = 0 \\ \text{Equations for } v : \quad & n_h^x + n_{h\gamma}^z = 0 \end{aligned}$$

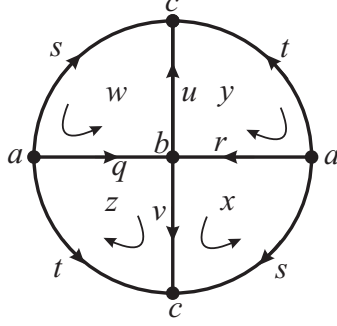
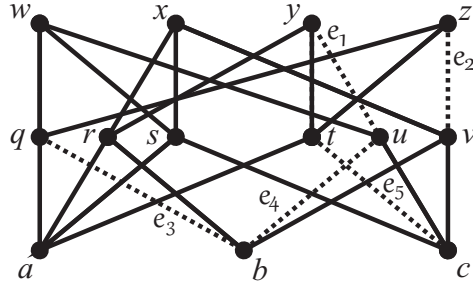


FIGURE 1. A regular CW-structure of the projective plane.

FIGURE 2. The coloring \hat{c} of $\mathcal{X}(K)$.

Therefore $\pi_2(K) = \{n(w - \gamma w - x + \gamma x + y - \gamma y - z + \gamma z) \mid n \in \mathbb{Z}\}$ is isomorphic to \mathbb{Z} . In fact K is just the real projective plane, so the results are not surprising. However, the example shows how to carry on the computation of π_2 for arbitrary regular CW-complexes.

Theorem 2.3. *Let K be a connected regular CW-complex of dimension 2 and let K' be its barycentric subdivision. Consider the full (one-dimensional) subcomplex L of K' spanned by the barycenters of the 1-cells and 2-cells. If the inclusion of each component of L in K' induces the trivial morphism between the fundamental groups, then $\pi_2(K) = \mathbb{Z}[\pi_1(K)] \otimes H_2(K)$.*

Proof. Let D be the subgraph of the Hasse diagram of $\mathcal{X}(K)$ induced by the points of height 1 and 2. Then L is the classifying space $\mathcal{K}(D)$ of the poset associated to D and the weak homotopy equivalence $\mu : K' \rightarrow \mathcal{X}(K)$ restricts to a weak equivalence $L \rightarrow D$. Moreover, for each component L_i of L , $\mu|_{L_i}$ is a weak equivalence between L_i and a component D_i of D . If $L_i \hookrightarrow K'$ induces the trivial map in π_1 , then so does the inclusion $D_i \hookrightarrow \mathcal{X}(K)$. By [3, Remark 4.2] each admissible G -coloring of $\mathcal{X}(K)$ is equivalent to another which is trivial in the edges of D . In particular, if c is a $\pi_1(K)$ -coloring which corresponds to the universal cover, then it is equivalent to a coloring c' such that $c'(x, y) = 1$ for each $(x, y) \in D$. Then $\tilde{X} = E(c')$ is also the universal cover of $\mathcal{X}(K)$ and

$$d = 1 \otimes \delta : C_2(\tilde{X}) = \mathbb{Z}[\pi_1(K)] \otimes C_2(\mathcal{X}(K)) \rightarrow C_1(\tilde{X}) = \mathbb{Z}[\pi_1(K)] \otimes C_1(\mathcal{X}(K)),$$

where $\delta : C_2(\mathcal{X}(K)) \rightarrow C_1(\mathcal{X}(K))$ is the boundary map of the chain complex associated to $\mathcal{X}(K)$. Since $\mathbb{Z}[\pi_1(K)]$ is a free \mathbb{Z} -module, $\pi_2(K) = H_2(\tilde{X}) = \mathbb{Z}[\pi_1(K)] \otimes H_2(K)$ by the Künneth formula. \square

When K is simply-connected, the previous result reduces to the Hurewicz Theorem for dimension 2.

Theorem 2.3 can be restated as follows: If every closed edge-path of K' containing no vertex of K is equivalent to the trivial edge-path, then $\pi_2(K) = \mathbb{Z}[\pi_1(K)] \otimes H_2(K)$.

There is an obvious generalization of Theorem 2.3 to connected regular CW-complexes with no restriction on the dimension.

Corollary 2.4. *Let K be a connected regular CW-complex. If every closed edge-path of K' containing only vertices which are barycenters of 1, 2 or 3-dimensional simplices is equivalent to the trivial edge-path, then $\pi_2(K) = \mathbb{Z}[\pi_1(K)] \otimes H_2(K)$.*

The following is another application of our methods (compare with [12]).

Theorem 2.5. *Let X and Y be two connected CW-complexes. If Y is simply-connected, then $\pi_2(X \vee Y) = \pi_2(X) \oplus (\mathbb{Z}[\pi_1(X)] \otimes \pi_2(Y))$.*

Proof. Since each CW-complex is homotopy equivalent to a simplicial complex, it suffices to prove the result for face posets X and Y of regular CW-complexes. Here, $X \vee Y$ denotes the space whose Hasse diagram is obtained from the diagrams of X and of Y by identifying a minimal element of each. Let c be a coloring of $X \vee Y$ corresponding to the universal cover. Then c is a G -coloring with $G \simeq \pi_1(X \vee Y) \simeq \pi_1(X)$. Since Y is simply-connected, there is an equivalent G -coloring c' which is trivial in Y (once again by Lemma 4.1 or Remark 4.2 in [3]). The restriction of c' to X is an admissible connected G -coloring. Moreover, if a closed edge-path in X is in $\ker(W_{c'|_X})$, then it is in $\ker(W_{c'}) = 0$. Thus, it is trivial in $\mathcal{H}(X \vee Y)$ and then in $\mathcal{H}(X)$, since the inclusion $X \hookrightarrow X \vee Y$ induces an isomorphism between the fundamental groups. Therefore, $c'|_X$ corresponds to the universal cover of X .

Let $\widetilde{X \vee Y} = E(c')$ be the universal cover of $X \vee Y$. Note that

$$C_n(\widetilde{X \vee Y}) = \mathbb{Z}[G] \otimes C_n(X \vee Y) = (\mathbb{Z}[G] \otimes C_n(X)) \oplus (\mathbb{Z}[G] \otimes C_n(Y))$$

for $n = 1, 2$. Since $c'|_X$ corresponds to the universal cover of X and $c'|_Y$ is trivial, the differential

$$d : (\mathbb{Z}[G] \otimes C_2(X)) \oplus (\mathbb{Z}[G] \otimes C_2(Y)) \rightarrow (\mathbb{Z}[G] \otimes C_1(X)) \oplus (\mathbb{Z}[G] \otimes C_1(Y))$$

has the form $d = d_{\widetilde{X}} \oplus (1_{\mathbb{Z}[G]} \otimes d_Y)$, where $d_{\widetilde{X}} : C_2(\widetilde{X}) \rightarrow C_1(\widetilde{X})$ is the differential in the chain complex associated to the universal cover of X and $d_Y : C_2(Y) \rightarrow C_1(Y)$ is the differential in the complex associated to Y . By the Künneth formula, $\pi_2(X \vee Y) = \ker(d) = H_2(\widetilde{X}) \oplus (\mathbb{Z}[G] \otimes H_2(Y)) = \pi_2(X) \oplus (\mathbb{Z}[G] \otimes \pi_2(Y))$. \square

3. RESULTS ON ASPHERICITY

We use the methods developed above to study asphericity of two-dimensional complexes and group presentations.

Theorem 3.1. *Let K be a 2-dimensional regular CW-complex and let K' be its barycentric subdivision. Consider the full (one-dimensional) subcomplex $L \subseteq K'$ spanned by the barycenters $b(\tau)$ of the 2-cells τ of K and the barycenters of the 1-cells which are faces of exactly two 2-cells. Suppose that for every connected component M of L , $i_*(\pi_1(M)) \leq \pi_1(K')$ contains an element of infinite order, where $i_* : \pi_1(M) \rightarrow \pi_1(K')$ is the map induced by the inclusion. Then K is aspherical.*

Proof. Let c be a G -coloring of $\mathcal{X}(K)$ which corresponds to the universal cover. We will use the equations describing $\pi_2(K)$ to show that if $\alpha = \sum_{h(x)=2} \sum_{g \in G} n_g^x gx \in \pi_2(K)$, then $n_g^x = 0$ for every $g \in G$ and every x with $h(x) = 2$. Let $x = \tau$ be a maximal element of $\mathcal{X}(K)$. Then $W = W_c : \mathcal{H}(\mathcal{X}(K), x) \rightarrow G$ is an isomorphism.

Let Y be the subspace of $\mathcal{X}(K)$ consisting of the 2-cells and the 1-cells which are faces of exactly two 2-cells. Note that $L = \mathcal{K}(Y)$, so there is a weak homotopy equivalence $L \rightarrow Y$. Since $i_*(\pi_1(L, b(\tau)))$ contains an element of infinite order and W is an isomorphism, there is a closed edge-path ξ at x in Y of weight $w(\xi) \in G$ of infinite order. We may assume that ξ is an edge-path of minimum length satisfying this property. Suppose ξ is the edge-path $x = x_0 \succ w_0 \prec x_1 \succ w_1 \prec \dots \succ w_{k-1} \prec x_k = x$. By the minimality of ξ , $x_{i+1} \neq x_i$ for every $0 \leq i < k$. Since x_i and x_{i+1} are the unique two elements covering w_i , the equation corresponding to w_i and an element $g \in G$ is

$$[x_i : w_i] n_{gc(w_i, x_i)}^{x_i} + [x_{i+1} : w_i] n_{gc(w_i, x_{i+1})}^{x_{i+1}} = 0.$$

In particular, given $g \in G$, if $n_g^{x_i} \neq 0$, then $n_{gc(w_i, x_i)^{-1}c(w_i, x_{i+1})}^{x_{i+1}} \neq 0$.

Let $h \in G$. Suppose that $n_h^x \neq 0$. Applying the previous assertion k times we obtain that $n_{hw(\xi)}^x \neq 0$. Repeating this reasoning we deduce that $n_{hw(\xi)^l}^x \neq 0$ for every $l \geq 0$. However, $w(\xi) \in G$ has infinite order and this contradicts the fact that only finitely many n_g^z can be non-zero. \square

Note that from the previous result one deduces the well-known fact that all compact surfaces different from S^2 and $\mathbb{R}P^2$ are aspherical. Any triangulation K of such surfaces satisfies the hypotheses of the theorem since every edge of K is face of exactly two 2-simplices and the links of the vertices are connected.

Example 3.2. The pinched two-handled torus and the wedge of two torii (Figure 3) are aspherical by Theorem 3.1.

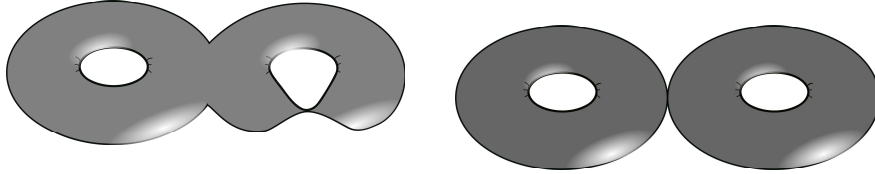


FIGURE 3. Aspherical two-complexes.

Remark 3.3. It is well-known that the fundamental group of any 2-dimensional aspherical complex is torsion-free (see [6, Proposition 2.45]). Theorem 3.1 says that if the 2-complex K has a torsion-free fundamental group and the maps $i_* : \pi_1(M) \rightarrow \pi_1(K')$ are non-trivial then K is aspherical.

We derive from Theorem 3.1 a result on asphericity of group presentations. This result resembles in some sense the homological description of π_2 using Reidemeister chains [10, Thm 3.8] (See also [5]). Given a group presentation P , let K_P be the usual two-dimensional

CW-complex associated to the presentation, which has one 0-cell, one 1-cell for each generator and one 2-cell for each relator. The presentation P is called aspherical if K_P is aspherical. In order to study asphericity of P , we will construct a digraph D_P associated to P together with a G -coloring. First note that the notion of a G -coloring naturally extends to directed graphs. A G -coloring of a digraph D is a labeling of the edges of D by elements in G . We allow loops and parallel edges which could have different colors. The color of the inverse of an edge e is the inverse $c(e)^{-1}$ of the color of e . A G -coloring c induces a weight map w_c . If $\alpha = e_0 e_1 \dots e_n$ is a cycle in the underlying undirected graph of D (for each i , e_i is an edge of D or e_i^{-1} is an edge of D), then $w_c(\alpha) = c(e_0)c(e_1) \dots c(e_n)$.

Let $P = \langle a_1, a_2, \dots, a_k \mid r_1, r_2, \dots, r_s \rangle$ be a presentation of a group G . The vertices of the directed graph D_P are the letters a_i which appear in total exactly twice in the words r_1, r_2, \dots, r_s . So, a_i appears either with exponent 2 or -2 in one of the relators and does not appear in any other relator, or it appears twice (in the same relator or in two different relators) with exponent 1 or -1 each time. Each vertex of D_P will be the source of exactly two oriented edges and the target of two directed edges. Let $r = r_j = a_{i_0}^{\epsilon_0} a_{i_1}^{\epsilon_1} \dots a_{i_{t-1}}^{\epsilon_{t-1}}$ be one of the relators of P , $\epsilon_l = \pm 1$ for every $l \in \mathbb{Z}_t$. We consider r as a cyclic word, so for example a_{i_1} comes after a_{i_0} and a_{i_0} comes after $a_{i_{t-1}}$. Suppose a_{i_l} is a vertex of D_P . We consider the first letter $a_{i_{l+m}}$ coming after a_{i_l} which is a vertex of D_P (i.e. the minimum $m > 0$ such that $a_{i_{l+m}} \in D_P$). It could be a letter different from a_{i_l} or the same letter if a_{i_l} appears twice in r or if it appears once and no other a_{i_s} is a vertex of D_P . Then $(a_{i_l}, a_{i_{l+m}})$ is a directed edge of D_P and the color corresponding to that edge is the subword $g a_{i_{l+1}}^{\epsilon_{l+1}} a_{i_{l+2}}^{\epsilon_{l+2}} \dots a_{i_{l+m-1}}^{\epsilon_{l+m-1}} h \in G$ where $g = 1$ if $\epsilon_l = 1$ and $g = a_{i_l}^{\epsilon_l}$ if $\epsilon_l = -1$, $h = 1$ if $\epsilon_{l+m} = -1$ and $h = a_{i_{l+m}}^{\epsilon_{l+m}}$ if $\epsilon_{l+m} = 1$.

The next example illustrates the situation.

Example 3.4. Figure 4 shows the digraph D_P corresponding to the presentation $P = \langle a, b, c, d, e \mid b^2 c a^{-1} b^{-1} d b a, c^{-1} d e b e \rangle$. Its vertices are a, c, d and e .

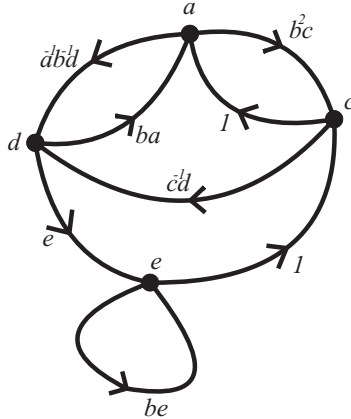


FIGURE 4. The digraph D_P associated to P .

Theorem 3.5. *Let P be a presentation of a group G . Suppose that every relator in P contains a letter which is a vertex of D_P . If each component of D_P contains a cycle whose weight has infinite order in G , then P is aspherical.*

Proof. We subdivide K_P barycentrically to obtain a regular CW-complex K as usual. Each 1-cell corresponding to a generator a in P is subdivided in two 1-cells e_{a_0} and e_{a_1} sharing the unique vertex v of K_P and a new vertex v_a . The 2-cell f_r corresponding to a relator r of P is subdivided in $2m$ 2-cells where m is the number of letters in r , adding a new 0-cell v_r in the interior of the original 2-cell. Let L be the 1-dimensional subcomplex of K' defined as in the statement of Theorem 3.1. The vertices of L are the barycenters of the 2-cells of K and the barycenters of the 1-cells which are faces of exactly two 2-cells. In the interior of the cell f_r there are exactly $4m$ vertices of L (the barycenters of the $2m$ 2-cells and the barycenters of the $2m$ edges from v_r to v and to each v_a). This 1-dimensional complex of $4m$ vertices is a cycle that we denote C_r . The remaining vertices of L are the barycenters $b(e_{a_0})$ and $b(e_{a_1})$ for each letter a which is a vertex of D_P . We show that the hypotheses of the theorem ensure that the hypotheses of Theorem 3.1 are fulfilled.

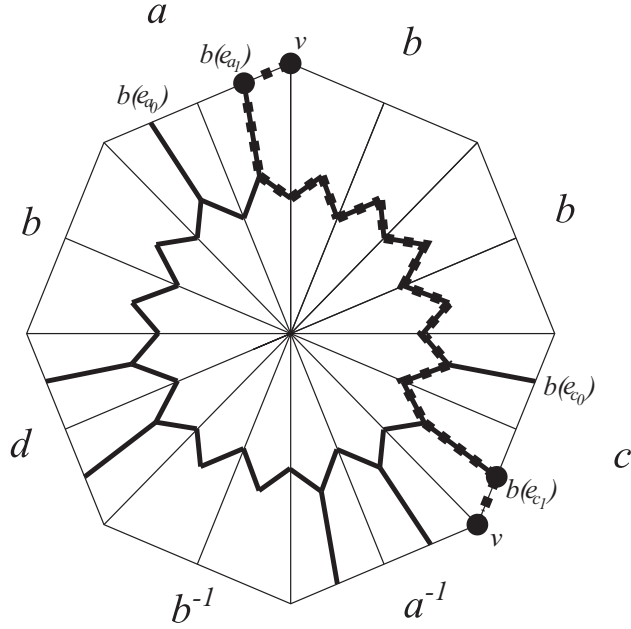


FIGURE 5. The 2-cell f_r . Here r is the first relator of P in Example 3.4. The simplices of the subcomplex L are drawn with thick lines and the edge-path γ_i homotopic to $b^2c \in G$ appears with dotted segments.

Since each relator contains a letter which is a vertex of D_P , the components of D_P are in bijection with the components of L . Suppose a and c are vertices of D_P and that there is an edge $(a, c) \in D_P$ (or (c, a)). Then, there is a relator r of P such that a and c are letters of r . Since $a, c \in D_P$, $b(e_{a_1})$ and $b(e_{c_1})$ are vertices of L and they lie in the 2-cell f_r of K_P corresponding to r . Moreover, there is an edge in L from $b(e_{a_1})$ to the cycle C_r and an edge from $b(e_{c_1})$ to C_r . Therefore there is an edge-path in L from $b(e_{a_1})$ to $b(e_{c_1})$ entirely contained in f_r (see Figure 5). A cycle α in D_P with base point a , has associated then a closed edge-path ξ in L at $b(e_{a_1})$. We will show that the order of ξ in the edge-path group $E(K', b(e_{a_1}))$ is infinite or, equivalently, that the order of $\hat{\xi} = (v, b(e_{a_1}))\xi(b(e_{a_1}), v) \in E(K', v)$ is infinite. The edge-path $\hat{\xi}'$ obtained from $\hat{\xi}$ by inserting the edge-paths $(b(e_{l_1}), v)(v, b(e_{l_1}))$ at each vertex $b(e_{l_1})$ (l a letter in α) is

equivalent to $\hat{\xi}$. Suppose $a = l^0, l^1, \dots, l^k = a$ are the vertices of α . The edge-path $\hat{\xi}'$ is a composition of closed edge-paths γ_i in K' at v , each of them contained in a 2-cell f_{r_i} . The edge-path γ_i , as an element of $\pi_1(K, v)$, is homotopic to a loop contained in the boundary of f_{r_i} which is, as an element of G , the color of the edge (l^i, l^{i+1}) in α . Thus, $\hat{\xi}' \in \pi_1(K, v) \simeq G$ coincides with the weight of α and the first one has infinite order provided the second one does. □

In Example 3.4 there is an edge from c to d with color $c^{-1}d$, an edge from a to d with color $a^{-1}b^{-1}d$ and an edge from a to c with color b^2c . Therefore, there is a cycle with base point c whose weight is $c^{-1}d(a^{-1}b^{-1}d)^{-1}b^2c = c^{-1}bab^2c \in G$. It is easy to verify that this element has infinite order, since $a + 3b$ clearly has infinite order in the abelianization $G/[G : G]$. Since D_P has a unique component and both relators of P have at least one letter in D_P , Theorem 3.5 applies. This shows that P is aspherical.

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DEPARTAMENTO DE MATEMÁTICA-IMAS, FCEyN, UNIVERSIDAD DE BUENOS AIRES, BUENOS AIRES, ARGENTINA

E-mail address: `jbarmak@dm.uba.ar`

E-mail address: `gminian@dm.uba.ar`